

Analytical Extension of the Frazer-Fulco Unitarity Relations for Isovector Nucleon Form Factors to the Complex t -Plane

M. I. Krivoruchenko

Institut für Theoretische Physik, Universität Tübingen, Auf der Morgenstelle 14

D-72076 Tübingen, Germany

and

Institute for Theoretical and Experimental Physics B.Chermushkinskaya, 25

117259 Moscow, Russia ¹⁾

Abstract

The unitarity relations for isovector nucleon form factors in the two-pion approximation for the intermediate states are analytically extended in a model independent way from the real half-axis $t > 4\mu^2$ (μ is the pion mass) to the whole complex t -plane. As a result, the explicit integral representations for the nucleon form factors are constructed in terms of the pion form factor and imaginary parts of the t -channel p -wave πN -scattering amplitudes at $t < 0$. New method based on the explicit integral representations is developed for numerical evaluation of the two-pion contribution to the isovector nucleon form factors. The results of different methods are compared and their efficiency is discussed.

1 Introduction

Recent years new precise experimental data were collected for the electromagnetic nucleon form factors at low and intermediate momentum transfers at spacelike and timelike regions [2]. The experimental progress stimulates development of the phenomenological models for the nucleon form factors.

The unitarity relations for isovector nucleon form factors, discovered by Frazer and Fulco [3], appeared to be the first fruitful attempt to provide field-theoretical framework for description of the electromagnetic nucleon form factors at low momentum transfers. Being combined with the requirements of the analyticity, relativistic invariance, and crossing symmetry of the πN -scattering amplitudes, the unitarity relations allow to evaluate the two-pion contribution to the absorptive parts of the nucleon form factors at the cut $(4\mu^2, +\infty)$ using experimental data on the $\pi\pi$ - and πN -scattering amplitudes only. Similar relations based on the unitarity are used in the analysis of the electromagnetic pion form factor [4]. The unitarity relations explain the origin of the vector-meson dominance (VMD) in the pion and nucleon form factors. The VMD models are in wide use in the modern calculations [5] - [11].

The differential approximation scheme for the analytical continuation of the πN -scattering amplitudes from the s -channel ($\pi N \rightarrow \pi N$) to the t -channel ($\pi\pi \rightarrow N\bar{N}$), suggested by Efremov, Meshcheryakov and Shirkov [12], was used by Lendel et al. [13] to evaluate the nucleon form factors from the unitarity relations (for a review see Ref. [14]).

Hhler and Pietarinen [15] performed the alternative analysis using the prescription by Frazer and Fulco [3] for calculation of the t -channel πN -scattering amplitudes at $t < 0$ and the discrepancy function method to extrapolate the amplitudes to the region $t > 4\mu^2$ (for a review see Ref. [16]).

The agreement of these two methods is surprisingly pure. The estimates of the two-pion contribution to the isovector nucleon form factor $F_{2v}(t=0)$ are in reasonable agreement, whereas the estimates of the form factor $F_{1v}(t=0)$ differ by a factor of two. There were no attempts to analyze the origin of this discrepancy. This might be a reason for the unitarity relations are simply ignored during the last two decades in the most calculations of the nucleon form factors.

The apparent disagreement of these two methods indicates presence of uncertainties

whose origin is not yet understood clearly. The purpose of this work is to discuss possible sources of the errors and develop new method free from ambiguities inherent in the earlier calculations of isovector nucleon form factors.

The unitarity relations by Frazer and Fulco are derived in the two-pion approximation for the intermediate states between the photon and the $N\bar{N}$ -pair (see Fig.1). Integral representations for the nucleon form factors, equivalent to the FF unitarity [3], are constructed in Refs. [17, 18] using the methods of Refs. [3], [12]-[14] and Refs. [3, 15, 16] and assuming the validity of the Frazer-Fulco-Gounaris-Sakurai (FFGS) model for the pion form factor [3, 19]. These representations extend the FF unitarity from the cut $(4\mu^2, +\infty)$ to the complex t -plane. For the kaon form factor, the representation of such a kind is constructed in Ref. [20]. In this paper, we extend the FF unitarity to the complex t -plane in a model independent way and give numerical estimates for the nucleon form factors, based on these representations.

The outline of the paper is as follows. In the next Sect., we construct analytical representations for the nucleon form factors valid in the complex t -plane, completely equivalent to the FF unitarity. In Sect.3, we develop new method for numerical evaluation of isovector nucleon form factors, based on these representations. In Sect.4, we compare our results with results of the previous works and discuss efficiency of different methods for evaluation of the two-pion contribution to the nucleon form factors from the unitarity.

2 Analytical Extension of the Unitarity Relations

The isovector nucleon form factors are defined by isovector part of the electromagnetic nucleon current

$$j_\mu(p', s', p, s)_v = \bar{u}(p', s')[F_{1v}(t)\gamma_\mu + \frac{1}{2m}F_{2v}(t)i\sigma_{\mu\nu}q_\nu]\tau_3 u(p, s) \quad (2.1)$$

where $t = (p' - p)^2$, m the nucleon mass, and τ_3 the third isotopic Pauli matrix. The form factors are normalized by $F_{1v}(0) = 1/2$ and $F_{2v}(0) = (\mu_p - \mu_n - 1)/2$ with μ_p and μ_n being the proton and neutron magnetic moments (in n.m.). The proton charge is set equal to unity.

The unitarity relations for nucleon form factors have the form [3]

$$ImF_{iv}(t) = -\frac{k^3}{\sqrt{t}}F^*(t)\Gamma_i(t) \quad (2.2)$$

where $k = \sqrt{t/4 - \mu^2}$, μ is the pion mass, and

$$\begin{aligned}\Gamma_1(t) &= \frac{m}{p_-^2}[-f_+^1 + \frac{t}{4\sqrt{2}m}f_-^1], \\ \Gamma_2(t) &= \frac{m}{p_-^2}[f_+^1 - \frac{m}{\sqrt{2}}f_-^1]\end{aligned}\quad (2.3)$$

with $p_-^2 = m^2 - t/4$. The $f_\pm^1(t)$ are the t -channel p -wave πN -scattering amplitudes. The unitarity relations are shown graphically in Fig. 1.

In the approximation of neglecting all but the two-pion intermediate state, the values $\Gamma_i(t)$ obey the dispersion relations [3]

$$\Gamma_i(t)/F_\pi(t) = \frac{1}{\pi} \int_{-\infty}^a \frac{Im\Gamma_i(t')}{t' - t} \frac{dt'}{F_\pi(t')} \quad (2.4)$$

with

$$a = 4\mu^2 - \frac{\mu^4}{m^2}. \quad (2.5)$$

Eqs.(2.2) and (2.4) are exactly valid below the four-pion threshold at the interval $4\mu^2 < t < 16\mu^2$. The branching ratio [21] $B(\rho \rightarrow 4\pi) < 0.002$ is small, so the two-pion approximation for the unitarity relations is valid with high accuracy at the ρ -meson peak and even at higher energies.

It is useful to define a function

$$K_\pi(t) = \frac{1}{\pi} \int_{4\mu^2}^{+\infty} \frac{t_1}{t_1 + t'} \frac{(t'/4 - \mu^2)^{3/2}}{\sqrt{t'}} |F_\pi(t')|^2 \frac{dt'}{t' - t}. \quad (2.6)$$

with $t_1 > 0$. This function is analytical in the complex t -plane with a cut $(4\mu^2, +\infty)$. The factor $t_1/(t_1 + t')$ is introduced to ensure better convergence of the integral. With the help of the function (2.6) the unitarity relations (2.2) take the form

$$ImF_{iv}(t) = -\frac{t_1 + t}{t_1} ImK_\pi(t) \Gamma_i(t)/F_\pi(t). \quad (2.7)$$

Using Eq.(2.4), the once subtracted dispersion integral for the nucleon form factors can easily be evaluated to give

$$F_{iv}(t) = F_{iv}(-t_1) + \frac{t_1 + t}{t_1} \frac{1}{\pi} \int_{-\infty}^a Im\Gamma_i(t') \frac{K_\pi(t) - K_\pi(t')}{t - t'} \frac{dt'}{F_\pi(t')}. \quad (2.8)$$

For computation of the nucleon form factors, it is sufficient to know the imaginary parts $Im\Gamma_i(t)$ at $t < a$. The integral in Eq.(2.8), however, converges very slowly. It is determined

by low- and high-energy parts of the values $Im\Gamma_i(t)$. The high-energy part $-\infty < t < b$ is unknown. It has been shown [22] that there are no objections against the truncated partial wave expansion in the region $-40\mu^2 < t < 0$, so we set $b = -40\mu^2$. The values $\Gamma_i(t)$ can be decomposed to the Born and rescattering parts: $\Gamma_i(t) = \Gamma_{iB}(t) + \Gamma_{iR}(t)$. The real and imaginary parts of the values $\Gamma_{iR}(t)$ are known on the interval $(b, 0)$ from the phase-shift analysis of the πN -scattering amplitudes. The problem consists in an approximate evaluation of the high-energy part with the help of the information on the low-energy real parts $Re\Gamma_i(t)$.

The once subtracted dispersion representation for the values $\Gamma_i(t)$ has the form

$$\Gamma_i(t)/F_\pi(t) = Re\frac{\Gamma_i(\kappa)}{F_\pi(\kappa)} + \frac{t - \kappa}{\pi} P \int_{-\infty}^a \frac{Im\Gamma_i(t')}{(t' - \kappa)(t' - t)} \frac{dt'}{F_\pi(t')}. \quad (2.9)$$

The value κ is assumed to be real, $\kappa < a$. The analog of Eq.(2.8) reads

$$F_{iv}(t) = F_{iv}(-t_1) + \frac{t_1 + t}{t_1} \left(Re\frac{\Gamma_i(\kappa)}{F_\pi(\kappa)} + \frac{1}{\pi} P \int_{-\infty}^a \frac{Im\Gamma_i(t')}{(t' - \kappa)} \frac{(t - \kappa)K_\pi(t) - (t' - \kappa)K_\pi(t')}{(t - t')} \frac{dt'}{F_\pi(t')} \right). \quad (2.10)$$

In comparison with Eq.(2.8), the integral at high t is suppressed by the additional power of t . Eqs.(2.8) and (2.10) constitute the main result of this section.

These equations can be simplified when the FFGS model [3, 19] for the pion form factor is applied. In this model, the pion form factor has the form

$$F_\pi(t) = \frac{t_0 + t}{t_0} \frac{D(0)}{D(t)} \quad (2.11)$$

where

$$D(t) = A + Bt + k^2 h(t) \quad (2.12)$$

is the D -function. It determines the phase shift $\delta_{11}(t)$ in the $I = J = 1$ $\pi\pi$ -partial wave. The exponentially large parameter $t_0 \propto \mu^2 \exp(\frac{8\pi k_\rho^3}{m_\rho^2 \Gamma_\rho})$ is a unique root of equation $D(-t_0) = 0$. Here m_ρ and Γ_ρ are the ρ -meson mass and width, $k_\rho = \sqrt{m_\rho^2/4 - \mu^2}$. The parameters A and B are fixed by requirements

$$\begin{aligned} \delta_{11}(m_\rho^2) &= \pi/2, \\ \delta'_{11}(m_\rho^2) &= 1/(m_\rho \Gamma_\rho). \end{aligned}$$

In this way one gets

$$A = m_\rho^2 + \mu^2 h_\rho + m_\rho^2 k_\rho^2 h'_\rho,$$

$$B = -1 - \frac{1}{4}h_\rho - k_\rho^2 h'_\rho. \quad (2.13)$$

The function $h(t)$ is determined at $t < 0$ by expression

$$\frac{k_\rho^3}{m_\rho^2 \Gamma_\rho} h(t) = \frac{1}{2\pi} \sqrt{\frac{4\mu^2 - t}{-t}} \ln\left(\frac{\sqrt{4\mu^2 - t} + \sqrt{-t}}{\sqrt{4\mu^2 - t} - \sqrt{-t}}\right). \quad (2.14)$$

The analytical continuation of this function to the complex t -plane has a cut at $t > 4\mu^2$. At the interval $0 < t < 4\mu^2$ the function $h(t)$ is analytical and real. At $t > 4\mu^2$

$$\begin{aligned} \text{Re}h(t) &= \frac{t - 4\mu^2}{t} h(t - 4\mu^2), \\ \frac{k_\rho^3}{m_\rho^2 \Gamma_\rho} \text{Im}h(t + i0) &= -\frac{k}{\sqrt{t}}. \end{aligned} \quad (2.15)$$

In Eqs.(2.7) $h_\rho = \text{Re}h(m_\rho^2)$ and $h'_\rho = \text{Re}h'(m_\rho^2)$. The D -function at the origin takes the form

$$D(0) = m_\rho^2 + \mu^2 h_\rho + m_\rho^2 k_\rho^2 h'_\rho - \frac{\mu^2 m_\rho^2 \Gamma_\rho}{\pi k_\rho^3}. \quad (2.16)$$

In the narrow width limit, $A = m_\rho^2$, $B = -1$, $D(0) = m_\rho^2$, $t_0 = \infty$, and so the $F_\pi(t)$ takes a monopole form.

The following relation takes place above the two-pion threshold

$$\frac{k^3}{\sqrt{t}} |F_\pi(t)|^2 = D(0) \frac{k_\rho^3}{m_\rho^2 \Gamma_\rho} \frac{t_0 + t}{t_0} \text{Im}F_\pi(t + i0). \quad (2.17)$$

The dispersion integral (2.6) for $t_1 = t_0$ can then be evaluated to give

$$K_\pi(t) = D(0) \frac{k_\rho^3}{m_\rho^2 \Gamma_\rho} F_\pi(t). \quad (2.18)$$

Eq.(2.10) becomes [18]

$$\begin{aligned} F_{iv}(t) &= F_{iv}(-t_0) - D(0) \frac{k_\rho^3}{m_\rho^2 \Gamma_\rho} \frac{t_0 + t}{t_0} F_\pi(t) \\ &+ \left(\text{Re} \frac{\Gamma_i(\kappa)}{F_\pi(\kappa)} + \frac{1}{\pi} P \int_{-\infty}^a \frac{\text{Im} \Gamma_i(t')}{(t' - \kappa)} \frac{(t - \kappa)/F_\pi(t') - (t' - \kappa)/F_\pi(t)}{(t' - t)} dt' \right). \end{aligned} \quad (2.19)$$

The imaginary part of the ratio $F_{iv}(t)/F_\pi(t)$ has a simple representation:

$$\text{Im}\{(F_{iv}(t) - F_{iv}(-t_0))/F_\pi(t)\} / \frac{k^3}{\sqrt{t}} = -\frac{1}{\pi} \int_{-\infty}^a \frac{\text{Im} \Gamma_i(t')}{t' - t} dt'. \quad (2.20)$$

3 Numerical Evaluation of the Nucleon Form Factors

In this Sect., we give numerical estimates for the nucleon form factors using the FFGS model for the pion form factor.

In the real part of the ratio $F_{iv}(t)/F_\pi(t)$, convergence of the integral can be improved by taking a weighted sum

$$Re\{(F_{iv}(t) - F_{iv}(-t_0))/F_\pi(t)\} = -D(0)\frac{k_\rho^3}{m_\rho^2\Gamma_\rho}\frac{t_0 + t}{t_0} \\ \sum_{j=1}^N c_{ij} \left(Re \frac{\Gamma_i(\kappa_j)}{F_\pi(\kappa_j)} + \frac{1}{\pi} P \int_{-\infty}^a \frac{Im\Gamma_i(t')}{(t' - \kappa_j)} \frac{(t - \kappa_j)/F_\pi(t') - (t' - \kappa_j)Re\{1/F_\pi(t)\}}{(t' - t)} dt' \right). \quad (3.1)$$

with the coefficients

$$\sum_{j=1}^N c_{ij} = 1. \quad (3.2)$$

The subtraction points κ_j are assumed to belong to the interval $b < \kappa_j < 0$, so the values $Re \frac{\Gamma_i(\kappa_j)}{F_\pi(\kappa_j)}$ entering Eq.(3.1) are known. The coefficients c_{ij} can be chosen such as to suppress the high-energy part $-\infty < t < b$ of the integral.

The integral in the right hand side of Eq.(3.1) can be rewritten in the form

$$\sum_{j=1}^N c_{ij} Re \frac{\Gamma_i(\kappa_j)}{F_\pi(\kappa_j)} + \frac{1}{\pi} \left(\int_{-\infty}^b + P \int_b^a \right) \sum_{j=1}^N c_{ij} Im\Gamma_i(t') \phi(t, t', \kappa_j) dt' \quad (3.3)$$

where

$$\phi(t, t', \kappa_j) = \frac{1}{t' - \kappa_j} \frac{(t - \kappa_j)/F_\pi(t') - (t' - \kappa_j)Re\{1/F_\pi(t)\}}{(t' - t)} \quad (3.4)$$

The first and third terms are known. We treat square of the second term as an error to be minimized. It can be evaluated as

$$\left(\frac{1}{\pi} \int_{-\infty}^b \sum_{j=1}^N c_{ij} Im\Gamma_i(t') \phi(t, t', \kappa_j) dt' \right)^2 \approx \left(\frac{1}{\pi} \int_{-\infty}^b \sum_{j=1}^N c_{ij} Im\Gamma_{iB}(t') \phi(t, t', \kappa_j) dt' \right)^2 \leq \\ \frac{1}{\pi^2} \int_{-\infty}^b (Im\Gamma_{iB}(t'))^2 dt' \int_{-\infty}^b \left(\sum_{j=1}^N c_{ij} \phi(t, t', \kappa_j) \right)^2 dt' = \epsilon_1^2 / \left(D(0) \frac{k_\rho^3}{m_\rho^2\Gamma_\rho} \right)^2 \quad (3.5)$$

The second source of the error is finite precision of the rescattering part of the values $Re\Gamma_{iR}(t)$. The corresponding contribution to the error of the expression (3.3) reads

$$\epsilon_2^2 = (D(0)\frac{k_\rho^3}{m_\rho^2\Gamma_\rho})^2 \sum_{j=1}^N c_{ij}^2 (Re\frac{\Gamma_{iR}(\kappa_j)}{F_\pi(\kappa_j)})^2 \Delta_j^2 \quad (3.6)$$

where $\Delta_j = 0.05$ are relative errors for the $Re\Gamma_{iR}(t)$. When N goes to infinity, the values Δ_j are no longer statistically independent, in which case expression (3.6) overestimates the error. In this work, we restrict ourselves by relatively small values $N \leq 12$. The errors of similar nature coming from the finite precision of the $Im\Gamma_{iR}(t)$ are neglected.

The conventional minimum of the total error $\epsilon_{tot}^2 = \epsilon_1^2 + \epsilon_2^2$ with respect to the coefficients c_{ij} under the constraint (3.2) imposed can be found by introducing the Lagrange multiplier λ . The coefficients c_{ij} and λ are solutions of a system of the $N+1$ linear equations obtained by taking derivatives with respect to the c_{ij} and λ of the function

$$\epsilon_1^2 + \epsilon_2^2 - \lambda(\sum_{j=1}^N c_{ij} - 1). \quad (3.7)$$

The coefficients c_{ij} and λ are t -dependent.

The real and imaginary parts of the rescattering amplitudes Γ_{iR} on the interval $(b, 0)$ are calculated in Ref. [15] and more recent results can be found in Ref. [16].

We test first convergence of the method with increasing the number N of the weight coefficients c_{ij} . In Figs. 2(a,b) the values of the form factors at zero momentum transfer are shown, together with the errors ϵ_{tot} , as functions of the number N . The subtractions are made at the points $\kappa_j = b(j-1/2)/N$. For $N > 6$ the errors decrease slowly. The predictions are more stable for the $F_{2v}(0)$. It is seen that variations of the $F_{1v}(0)$ are within the error bars. The variations of the $F_{2v}(0)$ are smaller than the theoretical errors. The coefficients c_{ij} have irregular behavior with increasing the N . The results shown in Figs. 2(a,b), however, agree with each other for different values of the N . This consistency check is equivalent to the consistency check for the analytical continuation of the amplitudes Γ_i to the region $t > 4\mu^2$ with the use of the discrepancy method.

The convergence of the integral in Eq.(2.20) for imaginary part of the ratio $F_{iv}(t)/F_\pi(t)$ can be improved by adding to the right hand side a weighted sum of expressions

$$Re\frac{\Gamma_i(\kappa_j)}{F_\pi(\kappa_j)} - \frac{1}{\pi}P \int_{-\infty}^a \frac{Im\Gamma_i(t')}{t' - \kappa_j} \frac{dt'}{F_\pi(t')} = 0. \quad (3.8)$$

The integrals in Eqs.(2.20) and (3.8) have similar structure. We chose the weights d_{ij} to suppress the high-energy part of the integral (2.20).

The right hand side of Eq.(2.20) takes the form

$$\sum_{j=1}^N d_{ij} Re \frac{\Gamma_i(\kappa_j)}{F_\pi(\kappa_j)} - \frac{1}{\pi} \left(\int_{-\infty}^b + P \int_b^a \right) Im \Gamma_i(t') \left(\sum_{j=1}^N d_{ij} \frac{1}{t' - \kappa_j} \frac{1}{F_\pi(t')} + \frac{1}{t' - t} \right) dt' \quad (3.9)$$

To ensure faster convergence of the integral, we require

$$\sum_{j=1}^N d_{ij} = 0. \quad (3.10)$$

The integral can be regularized afterwards by a substitution $1/(t' - \kappa_j) \rightarrow 1/(t' - \kappa_j) - 1/(t' - \kappa_0)$.

The error coming from the unknown high-energy part of the integral can be evaluated to be

$$\delta_1^2 = \frac{1}{\pi^2} \int_{-\infty}^b (Im \Gamma_{iB}(t'))^2 dt' \int_{-\infty}^b \left(\sum_{j=1}^N d_{ij} \frac{1}{(t' - \kappa_j)(t' - \kappa_0)} \frac{1}{F_\pi(t')} + \frac{1}{t' - t} \right)^2 dt' \quad (3.11)$$

The error coming from the finite precision of the rescattering parts $Re \Gamma_{iR}(t)$ reads

$$\delta_2^2 = \sum_{j=1}^N d_{ij}^2 \left(Re \frac{\Gamma_{iR}(\kappa_j)}{F_\pi(\kappa_j)} \right)^2 \Delta_j^2 \quad (3.12)$$

The conventional minimum of the total error $\delta_{tot}^2 = \delta_1^2 + \delta_2^2$ can be found with the help of the Lagrange multiplier γ corresponding to the constraint (3.10). The coefficients d_{ij} and γ are solutions of a system of the $N + 1$ linear equations obtained by setting equal to zero derivatives with respect to the d_{ij} and γ of the function

$$\delta_1^2 + \delta_2^2 - \gamma \sum_{j=1}^N d_{ij}. \quad (3.13)$$

In Figs. 2(c,d) the values $Im\{(F_{iv}(t) - F_{iv}(-t_0))/F_\pi(t)\}$ at the ρ -meson peak $t = m_\rho^2$, together with the errors, are shown as functions of the number N . The results seem to converge with increasing the N . The variations are within the error bars. The predictions are more stable for the $F_{2v}(m_\rho^2)$ than for the $F_{1v}(m_\rho^2)$. The variations of the $F_{2v}(m_\rho^2)$ are noticeably smaller as compared to the errors.

The theoretical errors are defined by replacing $\Gamma_i(t) \rightarrow \Gamma_{iB}(t)$ in the integral over the region $(-\infty, b)$. These estimates are valid up to unknown coefficients. These coefficients can apparently be fixed by requiring that the theoretical errors be equal to the statistical ones. In case of the form factor $F_{1v}(t)$, such a coefficient is close to unity, whereas in case of the form factor $F_{2v}(t)$ such a coefficient should be 5-7 times smaller than the unity.

Shown in Figs. 3(a-d) are real and imaginary parts of the ratios between the nucleon form factors and the pion form factor versus t . The errors ϵ_{tot} for the real parts and δ_{tot} for the imaginary parts are shown. The visible t -dependence of the errors δ_{tot} in Figs. 3(c,d) is mainly due to the logarithmic scale of the plots.

4 Discussions

In the differential approximation scheme by Efremov, Meshcheryakov and Shirkov [12], the t -channel p -wave projections of the πN -scattering amplitudes are calculated with an approximation of the p -wave amplitudes by linear superpositions of the forward and backward amplitudes and their higher derivatives. The analytical continuation to the unphysical region $t > 4\mu^2$ is performed with the help of dispersion relations for the forward and backward amplitudes. On the basis of the unitarity relations by Frazer and Fulco [3], the nucleon form factors are analyzed to the lowest order of the differential approximation by Lendel et al. [13] (see also Ref. [14]).

In the original work by Frazer and Fulco, the following program was suggested for evaluation of the nucleon form factors from the unitarity relations: The invariant s -channel πN -scattering amplitudes at fixed $t < 0$ are continued analytically from the physical region to the unphysical region between the s - and u -channels. Afterwards, the t -channel partial wave projections of these amplitudes can be computed. The complex p -waves $f_{\pm}^1(t)$ are known in the region $-26\mu^2 < t < 0$ in which the partial wave expansion converges. It has been shown [22] that there are no objections against the truncated expansion also in the region $-40\mu^2 < t < 0$. The analytical continuation of the complex p -waves $f_{\pm}^1(t)$ to the region $t > 4\mu^2$ can be made with the use of the extended unitarity relation, Eq.(2.4), according to which the ratios $f_{\pm}^1(t)/F_{\pi}(t)$ (equivalently $\Gamma_i(t)/F_{\pi}(t)$) are analytical functions in the complex t -plane with the left branch cut $(-\infty, a)$. In practice, the discrepancy function method is suitable in a restricted region of the complex t -plane for solving this task, since

the high-energy behavior of the scattering amplitudes is unknown. Given that the analytical continuation to the region $t > 4\mu^2$ is performed, one can compute the dispersion integral for the nucleon form factors. As t increases, the analytical continuation becomes less precise, so the upper limit $\Lambda^2 \approx 1 \text{ GeV}^2$ should be introduced to exclude the high-energy part of the dispersion integral. This program was performed, within the framework of the FFGS model, by Hoehler and Pietarinen [15] for the detailed analysis of the isovector nucleon form factors (see also Ref. [16]).

In the latter approach, one should compute three integrals numerically: The first one comes from the computation of the t -channel partial wave projections of the πN amplitudes at $t < 0$. The second one comes from the dispersion integral for analytical continuation of the p -wave amplitudes $f_{\pm}^1(t)$ to the region $t > 4\mu^2$. The third one comes from the dispersion integral for the nucleon form factors.

We demonstrated that the analytical evaluation of the last integral is also possible. In doing so, we arrived to the explicit one-dimensional integral representation (2.19) for the nucleon form factors in terms of the pion form factor and imaginary parts of the amplitudes $\Gamma_i(t)$ at $t < a$. There is no therefore need to perform analytical continuation of the amplitudes $\Gamma_i(t)$ to the region $t > 4\mu^2$. The representation (2.19) is constructed for the FFGS model. The model independent integral representations are given by Eqs.(2.8) and (2.10).

In Eq.(2.8), the imaginary part of the amplitudes $\Gamma_i(t)$ occurs only. Due to the extended unitarity (2.4), the low-energy real parts $Re\Gamma_i(t)$ are expressible through the high-energy imaginary parts $Im\Gamma_i(t)$. Therefore, the real parts contain information about high-energy behavior of the amplitudes. This information can be used for evaluation of the high-energy part of the integrals. In Sect.3, we described the corresponding numerical algorithm.

To get an idea about accuracy of the method [12] - [14], one needs to calculate the next order term in the differential expansion. This work is not done yet. It is clear, however, that the Born amplitude should be treated beyond the differential approximation method, since the logarithmic singularity at $t = a$ provides pure convergence of the series expansion.

The errors of the analysis [15] plotted on Fig.1 are ours. They are calculated using the claimed accuracy $\pm 15\%$ in Ref. [15] for the amplitudes $f_{\pm}^1(t)$ at the ρ -meson peak. There exist additional errors not plotted on Fig.2.

(i) The existence of the cut-off parameter $\Lambda^2 \approx 1 \text{ GeV}^2$ implies that a contribution of the ρ -meson tail $t > \Lambda^2$ to the nucleon form factors is disregarded. The results [15] are Λ -dependent. In our approach, the cut-off parameter is safely sent to infinity (the integrals converge).

(ii) In the analysis [15], the contribution from the interval $-120\mu^2 < t < b$, in which the truncated partial wave expansion of the amplitudes diverges, is included in the dispersion integral. The partial wave expansion gives at the interval $-120\mu^2 < t < b$ substantial part of the discontinuity $\text{Im}\Gamma_i(t)$. The corresponding errors cannot be controlled, since the values $\text{Im}\Gamma_{iR}(t)$ calculated from the partial wave expansion exhibit at $t < b$ strong tendency to increase with decreasing t .

The contribution from the interval $(-\infty, -120\mu^2)$ in Ref. [15] is interpolated by a polynomial $a + bt$. From the unitarity it follows that $\Gamma_1(t) = O(1/t)$ and $\Gamma_2(t) = O(1/t^{3/2})$ as $t \mapsto \infty$. Taking into account that in the FFGS model $F_\pi(t) = O(1/\log(t))$, we conclude that the dispersion integral in Eq.(2.4) can be evaluated, respectively, as $O(\log^2(t)/t)$ and $O(1/t)$ as $t \mapsto \infty$. The contribution from the interval $(-\infty, -120\mu^2)$ therefore vanishes when t increases. The polynomial representation $a + bt$ does not satisfy this requirement. The simple interpolations $a/(b+t)$ and $\log((a+t)/(b+t))$ could provide the correct asymptotic form.

The distinction between the methods [12] - [16] and ours makes comparison of the results not so straightforward.

The comparison of the results makes sense for zero value of the subtraction constant $F_{iv}(-t_0) = 0$. The quark counting rules imply that the total nucleon form factors, containing all multipion and $N\bar{N}$ contributions to the nucleon spectral functions, vanish as t goes to infinity. In the two-pion approximation, the value $F_{iv}(-t_0)$ should not, however, be strictly equal to zero.

In Fig. 2(a-d), we show our results at two points $t = 0$ and $t = m_\rho^2$ for different numbers of the weight coefficients. The errors are theoretical ones, ϵ_{tot} and $(k^3/\sqrt{t})\delta_{tot}$. The results of Ref. [15] are also displayed.

The most typical results are summarized in Table 1. Ref. [15] gives for the amplitudes $f_\pm^1(t)$ at the ρ -meson peak an error of $\pm 15\%$, resulting in turn in a small error of the form factor $F_{1v}(0)$, which is 4-5 times smaller as compared to our predictions. As we already

discussed, some of the uncertainties in the analysis of Ref. [15] can hardly be controlled, so the errors of Ref. [15] are greater as compared to those shown on Figs. 2 and 3 and in Table 1. Our method for large N gives systematically lower (negative) values for the difference $F_{1v}(0) - F_{1v}(-t_0)$. The error ϵ_{tot} is, however, large. The unitarity relations, therefore, do not allow to give accurate predictions for the form factor $F_{1v}(t)$. The value $F_{2v}(0)$ in our approach is well defined. The estimates of Refs. [13] and [15] are in reasonable agreement with ours. The accuracy of our predictions for the $F_{2v}(t)$ is better as compared to the earlier ones.

5 Concluding Remarks

The unitarity relations by Frazer and Fulco are written in the two-pion approximation for imaginary parts of the isovector nucleon form factors at the branch cut $(4\mu^2, +\infty)$. We showed that the unitarity relations for the nucleon form factors can be solved to give analytical one-dimensional integral representations for the nucleon form factors valid in the whole complex t -plane (Eqs.(2.13) and (2.15)). The nucleon form factors are expressed in terms of the pion form factor and the t -channel p -wave amplitudes $\Gamma_i(t)$ at $t < a$.

The representation (2.15) is used to develop new method for numerical evaluation of the nucleon form factors from the unitarity relations. The method is described in Sect.3. It has some advantages as compared to the earlier methods: (i) there is no need to perform analytical continuation of the amplitudes $\Gamma_i(t)$ from the region $t < 0$ to the region $t > 4\mu^2$, (ii) no cut off parameter Λ to be introduced in the dispersion integrals for the nucleon form factors.

The numerical estimates of the nucleon form factors from the unitarity relations in the two-pion approximation are at present in the rough qualitative agreement for different methods. These estimates are considered as efficient and reliable if results of the different methods will be brought into the correspondence by evaluation of the high-order approximations and careful analysis of possible sources of errors.

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References

- [1] Permanent address
- [2] A.Lung et al., Phys.Rev.Lett. **70** (1993) 718;
S.Rock et al., Phys.Rev.Lett. **49** (1993) 1139;
T.A.Armstrong et al., Phys.Rev.Lett. **70** (1993) 1212;
A.F.Still et al., Phys.Rev. **D48** (1993) 29;
L.Andivahis et al., Phys.Rev. **D50** (1994) 5491;
R.C.Walker et al., Phys.Rev. **D49** (1994) 5671;
S.Platchkov et al., Nucl.Phys. **A 510** (1990) 740;
T.Eden et al., Phys.Rev. **C 50** (1994) R1749;
M.Meyerhoff et al., Phys.Lett. **B 327** (1994) 201;
E.E.Bruns et al., Phys.Rev.Lett. **75** (1995) 21.
- [3] W.R.Frazer and J.Fulco, Phys.Rev.Lett. **2** (1959) 365;
Phys.Rev. **117** (1960) 1603; 1609.
- [4] J.D.Bjorken and S.D.Drell, *Relativistic Quantum Fields*
(McGraw-Hill, New-York, 1965).
- [5] G.Hoehler et al., Nucl.Phys. **114** (1976) 505.
- [6] S.Dubnicka, Nuovo Cim. **104A** (1991) 1075;
A.Z.Dubnickova and S.Dubnicka, Phys.Lett. **B261** (1991) 127.
- [7] P.Kroll, M.Schurmann and W.Schweiger, Z.Phys. **A338** (1991) 339.
- [8] M.Gari and W.Krumpelmann, Phys.Rev. **D45** (1992) 1817.
- [9] S.Furuichi and K.Watanabe, Progr.Th.Phys. **92** (1994) 339.
- [10] M.M.Giannini, E.Santopinto and M.I.Krivoruchenko, *Proc. Int. Conf. "Mesons and Nuclei at Intermediate Energies"*, Dubna, May 3 - 7, 1994.
- [11] P.Mergel, Ulf-G. Meissner and D.Drechsel, Nucl.Phys. **A596** (1996) 367.
- [12] A.V.Efremov, V.A.Meshcheryakov and D.V.Shirkov, JETP **39** (1960) 438.
- [13] A.I.Lendel, V.I.Lendyel, V.A.Meshcheryakov and B.M.Ernst, Yad.Fiz.**3** (1966) 1093.

- [14] D.V.Shirkov, V.V.Serebryakov and V.A.Meshcheryakov, *Dispersion Theories of Strong Interactions at Low Energies*, 1967 (Moscow, Nauka).
- [15] G.Hoehler and E.Pietarinen, Nucl.Phys. **B95** (1975) 210.
- [16] G.Hoehler, *Pion-Nucleon Scattering*, 1983 (Springer, Landolt-Börnstein), Vol.I/9b, ed. H.Schopper.
- [17] M.I.Krivoruchenko, Yad.Fiz. **58** (1995) 1839.
- [18] M.I.Krivoruchenko, πN Newsletter **10** (1995) 156.
- [19] G.Gounaris and J.J.Sakurai, Phys.Rev.Lett. **21** (1968) 244.
- [20] M.I.Krivoruchenko, Pis'ma ZhETF **58** (1993) 7; Z.Phys. **A350** (1995) 343.
- [21] Particle Data Group, Phys.Rev. **D54** (1996).
- [22] E.Pietarinen, Helsinki Preprint 17-76.

FIGURE CAPTIONS

Fig. 1. Pictorial representation of the unitarity relations for isovector nucleon form factors in the two-pion approximation for the intermediate states between the photon and the nucleon-antinucleon pair. The imaginary parts of the nucleon form factors are expressible in terms of a product of the pion form factor and the t -channel πN -scattering amplitudes.

Fig. 2. (a,b) The values $F_{iv}(0) - F_{iv}(-t_0)$ (left scale) versus the number N of the weight coefficients c_{ij} fixed by a requirement of minimum of the total errors ϵ_{tot} originating from the unknown high-energy part of the integrals and finite precision of the real parts of the t -channel p -wave scattering amplitudes. (c,d) The ratios $Im\{(F_{iv}(m_\rho^2) - F_{iv}(-t_0))/F_\pi(m_\rho^2)\}$ (right scale) at the ρ -meson peak versus the number N of the weight coefficients d_{ij} fixed by a requirement of minimum of the total errors δ_{tot} originating from the unknown high-energy part of the integral in Eq.(3.9) and finite precision of the real part of the t -channel p -wave scattering amplitudes. The results of Ref. [15] (dashed regions) are plotted for comparison.

Fig. 3. (a,b). Real part of the ratios between the nucleon form factors and the pion form factor, $Re\{(F_{iv}(t) - F_{iv}(-t_0))/F_\pi(t)\}$, versus t with errors ϵ_{tot} for $N = 6$ (left scale). (a,b) Imaginary part of the ratios between the nucleon form factors and the pion form factor, $Im\{(F_{iv}(t) - F_{iv}(-t_0))/F_\pi(t)\}/\frac{k^3}{\sqrt{t}}$, versus t with errors δ_{tot} for $N = 6$ (right scale). The results of Ref. [15] (dashed regions) are plotted for comparison.

TABLE 1. Comparison of predictions of different methods at two points $t = 0$ and $t = m_\rho^2$. In Ref. [13], the estimates are obtained in the first order of the differential approximation. The errors (presently unknown) can be estimated by computing the next order of the differential approximation. The errors of Ref. [15] are evaluated from the claimed accuracy $\pm 15\%$ for the amplitudes $f_\pm^1(m_\rho^2)$. Two sets of values are given on the basis of our calculations: shown in the line (I) are values corresponding to the maximum number of the weight coefficients $N = 12$. In the line (II) predictions obtained by statistical averaging the data from $N = 3$ to 12 are given. In the line (I), the errors are theoretical ones, ϵ_{tot} and $(k^3/\sqrt{t})\delta_{tot}$ (see the text), while in the lower line the errors are statistical ones. The reduction of the errors for the case of the form factor $F_{2v}(t)$ occurs, since the variation of the results for different N in this case is significantly smaller than the theoretical errors, ϵ_{tot} and $(k^3/\sqrt{t})\delta_{tot}$.

<i>Ref.</i>	$(F_{iv}(0) - F_{iv}(-t_0))/F_\pi(0)$		$Im\{(F_{iv}(m_\rho^2) - F_{iv}(-t_0))/F_\pi(m_\rho^2)\}$	
	$i = 1$	$i = 2$	$i = 1$	$i = 2$
[13]	1.15	3.09	0.05	0.09
[15]	0.52 ± 0.08	3.15 ± 0.57	0.20 ± 0.09	1.00 ± 0.21
<i>This work (I)</i>	-0.36 ± 0.35	2.73 ± 0.50	0.39 ± 0.07	1.08 ± 0.07
<i>This work (II)</i>	-0.03 ± 0.37	2.78 ± 0.12	0.34 ± 0.05	1.07 ± 0.01